

A NOTE ON NON-REDUCED PICARD SCHEMES

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ABSTRACT. The Picard scheme of a smooth curve and a smooth complex variety is reduced. In this note we discuss which classes of surfaces in terms of the Enriques–Kodaira classification can have non-reduced Picard schemes and whether there are restrictions on the characteristic of the ground field. It turns out that non-reduced Picard schemes are uncommon in Kodaira dimension $\kappa \leq 0$, that this phenomenon can be bounded for $\kappa = 2$ (general type) and that it is as bad as can be in $\kappa = 1$.

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INTRODUCTION

The set of isomorphism classes of invertible sheaves on a scheme X forms a group, the so-called Picard group $\mathrm{Pic}(X)$. In case X is integral and projective over a field k , this group $\mathrm{Pic}(X)$ carries a natural scheme structure as was shown by Grothendieck [Gr]. Moreover, if X is geometrically normal, then $\mathrm{Pic}^0(X)$, the identity component of $\mathrm{Pic}(X)$, is even projective.

A theorem of Cartier states that group schemes over fields of characteristic zero are reduced. It follows that Pic^0 of a projective and geometrically normal scheme is an Abelian variety in this case.

However, over fields of positive characteristic, the Pic^0 even of a smooth projective variety need no longer be reduced. A first example has been constructed by Igusa [Ig]. As explained by Mumford in [Mum, Lecture 27], the non-reducedness of the Picard scheme can be related to Bockstein operations in cohomology. It follows that varieties with $h^2(X, \mathcal{O}_X) = 0$ have a reduced Picard scheme. And in particular, Pic^0 of a geometrically normal curve is always an Abelian variety.

Hence we have to look at dimension two and in view of the Enriques–Kodaira classification it is natural to ask:

- (1) What kind of surfaces, e.g. ruled, elliptic, or general type, have a non-reduced Pic^0 ?
- (2) Fixing numerical invariants, is it true that surfaces with these invariants have a reduced Pic^0 ?
- (3) If the previous question has a negative answer in general, does it have a positive answer if the characteristic of the ground field is sufficiently large?

From the Enriques–Kodaira classification and its extension to positive characteristic by Bombieri–Mumford [BM1] we immediately get

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Proposition. *For Kodaira dimension $\kappa(X) \leq 0$, the Picard scheme tends to be reduced:*

- (1) *If $\kappa(X) = -\infty$ then $\text{Pic}^0(X)$ is reduced.*
- (2) *If $\kappa(X) = 0$ then $\text{Pic}^0(X)$ is reduced except for a few exceptional cases in characteristic 2 and 3.*

In Kodaira dimension $\kappa = 1$ all surfaces possess elliptic fibrations and the non-reducedness of the Picard scheme is closely related to the existence of wild fibres. Using results on torsors under Jacobian fibrations we show the following, which is more or less implicit in the literature:

Theorem. *Let $f : X \rightarrow B$ be an elliptic fibration of a surface in positive characteristic. Assume that f is not generically constant. Then there exists an elliptic fibration $f' : X' \rightarrow B$ such that*

- (1) *$\text{Pic}^0(X')$ not reduced,*
- (2) *$\kappa(X') = 1$.*
- (3) *the Jacobian fibrations of f and f' coincide, and*
- (4) *$b_i(X) = b_i(X')$ for all i and $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X'})$.*

In particular, for every positive characteristic and every set of Betti-numbers, Euler characteristic and not generically constant elliptic fibration for which there exists a surface with $\kappa = 1$, there exists a surface with the same invariants and a non-reduced Picard scheme.

Moreover, we can choose the difference between h^{01} and $\frac{1}{2}b_1$, which can be viewed as a measure of the non-reducedness of Pic^0 , as large as we want to.

Examples of Katsura and Ueno show that the situation is similarly bad for iso-trivial fibrations.

For Kodaira dimension $\kappa = 2$, i.e., surfaces of general type, there are examples due to Serre with non-reduced Picard schemes in every characteristic. However, we can limit this phenomenon

Theorem. *Given an integer m , there exists an integer $P(m)$, such that minimal surfaces of general type with $K_X^2 = m$ over fields of characteristic $p \geq P(m)$ have a reduced Pic^0 .*

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1. KODAIRA DIMENSION AT MOST ZERO

Let X be a smooth projective surface over an algebraically closed field k . We denote by $\kappa(X)$ its Kodaira dimension. Thanks to the Enriques–Kodaira classification that was extended to positive characteristic by Bombieri and Mumford we have an explicit description of surfaces with $\kappa(X) \leq 0$, which allows us to answer the questions posed in the introduction quite satisfactory.

Two smooth projective surfaces that are birational are related by a sequence of blow-ups and blow-downs in closed points. Since this process does not affect Pic^0 , we may and will restrict ourselves to suitable minimal models in the following.

Theorem 1.1. *If $\kappa(X) = -\infty$ then $\text{Pic}^0(X)$ is reduced.*

PROOF. A surface with $\kappa(X) = -\infty$ is birational to $\mathbb{P}^1 \times C$, where C is a smooth curve. Hence such a surface has a reduced Pic^0 . \square

Theorem 1.2. *If $\kappa(X) = 0$ then $\text{Pic}^0(X)$ is reduced except possibly if X is*

- (1) *a non-classical Enriques surface in characteristic 2, or*
- (2) *a (quasi-) hyperelliptic surface in characteristic 2 or 3.*

The exceptions do occur.

PROOF. A look at the table of possible invariants in the introduction of [BM1] shows that the only surfaces with non-reduced Pic^0 (noted as $\Delta \neq 0$ in this table) are non-classical Enriques surfaces or certain (quasi-)hyperelliptic surfaces.

Non-classical Enriques surfaces can exist in characteristic 2 only [BM1, Theorem 5] and such surfaces have been constructed in [BM2, Section 3].

Hyperelliptic surfaces of Kodaira dimension zero with non-reduced Pic^0 are those with $p_g = 1$, using the table of possible invariants again. These are precisely the hyperelliptic surfaces where K_X is of order 1, and the detailed analysis in [BM1, Section 3] shows that such surfaces can and do exist in characteristic 2 and 3 only.

Quasi-hyperelliptic surfaces exist in characteristic 2, 3 only [BM2]. As explained in the proof of [BM2, Proposition 8], such a surface has $\text{ord} K_X = 1$, i.e., a non-reduced Pic^0 , if and only if the character $K \rightarrow \text{Aut}(C_0)/G_a \cdot A \cong \mathbb{G}_m$ is trivial (notation as in loc.cit.). In characteristic 2, this condition is fulfilled for surfaces of type f) of the *Char.* 2-table in [BM2, page 214]. In characteristic 3, this condition holds for surfaces of type d) of the *Char.* 3-table in [BM2, page 214], cf. also [La, Section 3B]. \square

2. ELLIPTIC FIBRATIONS

We have seen in the first section that surfaces with $\kappa \leq 0$ and non-reduced Pic^0 form a very small class. This is not true in Kodaira dimension $\kappa = 1$, even when fixing numerical invariants. Since all these surfaces are endowed with an elliptic fibration we translate our problem into the language of elliptic fibrations. In fact, twisting an elliptic fibration and adding wild fibres we can make its Pic^0 as non-reduced as we want to whilst fixing numerical invariants.

We recall that $H^1(\mathcal{O}_X)$ can be identified with the Zariski tangent space to $\text{Pic}^0(X)$ and that $\frac{1}{2}b_1(X)$ is the dimension of $\text{Pic}^0(X)$. Hence the difference $h^{01} - \frac{1}{2}b_1$ can be viewed as a measure for the non-reducedness of Pic^0 , which is zero if and only if Pic^0 is reduced.

Let $f : X \rightarrow B$ be an elliptic fibration over a curve. We recall that a fibre F is called *wild*, if $h^0(F, \mathcal{O}_F) \geq 2$. Wild fibres can exist over fields of positive characteristic only and we refer to [CD, Chapter V] for details. The following result explains the role of wild fibres in view of non-reduced Picard schemes.

Proposition 2.1. *Let $f : X \rightarrow B$ be a relatively minimal elliptic fibration over a curve B .*

- (1) *If $\chi(\mathcal{O}_X) \geq 1$ and f has no wild fibres then $\text{Pic}^0(X)$ is reduced.*
- (2) *If f has $w \geq 2$ wild fibres then $\text{Pic}^0(X)$ is not reduced and $h^{01} - \frac{1}{2}b_1 \geq (w - 1)$.*

PROOF. We have $R^1 f_* \mathcal{O}_X \cong \mathcal{L} \oplus \mathcal{T}$, where \mathcal{L} is a line bundle on B and \mathcal{T} is a torsion sheaf whose support consists precisely of those points over which the fibre of f is wild. From the Grothendieck–Leray spectral sequence we obtain a short exact sequence

$$(1) \quad 0 \rightarrow H^1(B, \mathcal{O}_B) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^0(B, R^1 f_* \mathcal{O}_X) \rightarrow 0.$$

Assume $\chi(\mathcal{O}_X) \geq 1$ and that f has no wild fibres. Then $h^0(\mathcal{T}) = 0$ and the canonical bundle formula for elliptic fibrations gives $\deg \mathcal{L} = -\chi(\mathcal{O}_X) \leq -1$, hence $h^0(B, \mathcal{L}) = 0$. We obtain $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_B)$. By its universal property, the composition $X \rightarrow B \rightarrow \text{Jac}(B)$ factors over the Albanese variety of X , from which we conclude $b_1(X) \geq b_1(B) = 2h^1(\mathcal{O}_B) = 2h^1(\mathcal{O}_X)$. Since we have $b_1(X) \leq 2h^1(\mathcal{O}_X)$ in any case, we obtain $2h^1(\mathcal{O}_X) = b_1(X)$, which implies that $\text{Pic}^0(X)$ is reduced.

Now, assume that f has $w \geq 2$ wild fibres. Then $h^0(\mathcal{T}) \geq w$ and hence $h^1(\mathcal{O}_X) - h^1(\mathcal{O}_B) \geq w$ by (1). By [K-U, Lemma 3.4], we have $\frac{1}{2}b_1(X) \leq h^1(\mathcal{O}_B) + 1$, which yields the desired inequality. Since h^{01} is strictly larger than $\frac{1}{2}b_1$, the $\text{Pic}^0(X)$ is non-reduced. \square

The next result tells us that, given an elliptic surface in positive characteristic that is not generically constant, we can always find another fibration with $\kappa = 1$ and with the same Betti numbers but with arbitrary non-reduced Picard scheme. In particular, we cannot bound the non-reducedness by fixing invariants or the characteristic.

Theorem 2.2. *Let $f : X \rightarrow B$ be a relatively minimal elliptic fibration over a curve B defined over an algebraically closed field of positive characteristic. Let $n \geq 1$ be an integer and assume that f is not generically constant.*

Then there exists an elliptic fibration $f' : X' \rightarrow B$, such that

- (1) $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X'})$, $K_X^2 = K_{X'}^2 = 0$ and $b_i(X) = b_i(X')$ for all i ,
- (2) *both elliptic fibrations have the same Jacobian fibration,*
- (3) $\text{Pic}^0(X')$ *is non-reduced and even $h^{01} - \frac{1}{2}b_1 \geq n$, and*
- (4) $\kappa(X') = 1$ *if $n \geq 2$ or $p \geq 5$.*

PROOF. We use the notation of [CD, Section 5.4]. We denote by $j : J \rightarrow B$ the Jacobian fibration associated to $f : X \rightarrow B$. Let $J_\eta^\#$ be the Néron model of the generic fibre of j . We denote by $\text{Elf}(j)$ the abelian group classifying torsors under $J_\eta^\#$. For every closed point $b \in B$, we let $\tilde{\mathcal{O}}_{B,b}$ be the (strict) Henselisation of the local ring $\mathcal{O}_{B,b}$. Let $\tilde{J}_b^\#$ be the Néron model of (the reduction) of $J \times_B \text{Spec } \tilde{\mathcal{O}}_{B,b}$ and let $\text{Elf}(\tilde{j}_b)$ be the abelian group of torsors under $\tilde{J}_b^\#$. For every closed point $b \in B$ there exists a homomorphism $\psi_b : \text{Elf}(j) \rightarrow \text{Elf}(\tilde{j}_b)$, the so-called *local invariant*.

Since f is not generically constant, j is not trivial and in this case there exists a short exact sequence

$$(2) \quad 0 \rightarrow \text{III}(J_\eta^\#) \rightarrow \text{Elf}(j) \xrightarrow{\psi} \bigoplus_{b \in B} \text{Elf}(\tilde{j}_b) \rightarrow 0, \quad \text{where} \quad \psi = \sum_{b \in B} \psi_b,$$

cf. [CD, Proposition 5.4.3] and [CD, Corollary 5.4.6].

The generic fibre of j is an ordinary elliptic curve as j is not trivial. If the fibre above b is an ordinary elliptic curve, there exists a non-trivial subgroup $\text{Elf}(\tilde{j}_b)^{\text{rad}}$ of $\text{Elf}(\tilde{j}_b)$, such that an element of $\text{Elf}(j)$, which maps to a non-trivial element of $\text{Elf}(\tilde{j}_b)^{\text{rad}}$ corresponds to an elliptic fibration with Jacobian fibration j and a wild fibre above b , cf. [CD, Corollary 5.4.3].

We choose a set S of $(n+1)$ distinct points in B such that the fibres of j above these points are ordinary elliptic curves. For every $b \in S$ we choose a non-trivial element e_b in $\text{Elf}(\tilde{j}_b)^{\text{rad}}$. By the surjectivity of ψ in (2), there exists an element f' of $\text{Elf}(j)$ such that $\psi_b(f') = e_b$ for every $b \in S$. This f' corresponds to an elliptic fibration $f' : X' \rightarrow B$ with wild fibres above S . By Proposition 2.1, we have $h^{01} - \frac{1}{2}b_1 \geq n$ and that $\text{Pic}^0(X')$ is not reduced.

By [CD, Proposition 5.3.6] we have $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_J) = \chi(\mathcal{O}_{X'})$ and the same for the Betti numbers and c_2 by [CD, Corollary 5.3.5]. We have $K^2 = 0$ in any case.

If $h^{01} - \frac{1}{2}b_1 \geq n \geq 1$ then $\kappa(X') \geq 0$ by Theorem 1.1. By the table of possible invariants in the introduction of [BM1], we see that $\kappa(X') = 0$ and $n \geq 1$ implies $h^{01} - \frac{1}{2}b_1 = 1$ and $p \leq 3$. Hence if $n \geq 2$ or $p \geq 5$ we have $\kappa(X') = 1$. \square

Even among iso-trivial elliptic surfaces with $\kappa = 1$ we find arbitrary non-reduced Picard schemes in arbitrary positive characteristic. The following examples are due to Katsura and Ueno:

Proposition 2.3. *For every prime p and every integer n there exists an elliptic surface with $\kappa = 1$ defined over an algebraically closed field of characteristic p such that*

- (1) *the elliptic fibration is iso-trivial*
- (2) Pic^0 *is not reduced and even* $h^{01} - \frac{1}{2}b_1 \geq n$.

PROOF. Let X be an elliptic surface of [K-U, Example 8.1]. As X possesses an iso-trivial elliptic fibration, we have $\chi(\mathcal{O}_X) = 0$. By [K-U, Lemma 3.5] we have $b_1 = 2$. For $m \geq 3$ (as defined in [K-U, Example 8.1]) we have $\kappa(X) = 1$ and choosing m sufficiently large, we get p_g as large as we want to, i.e., we also get h^{01} as large as we want to since $\chi(\mathcal{O}_X) = 0$. \square

3. GENERAL TYPE

There exist surfaces with $\kappa = 2$, i.e., surfaces of general type, with non-reduced Picard schemes in arbitrary large characteristic. However, fixing K_X^2 , there exists only a finite number of characteristics where minimal surfaces of general type with these invariants can have non-reduced Picard schemes.

We recall that surfaces of general type can have non-reduced Picard schemes in arbitrary large characteristic - the examples are due to Serre:

Proposition 3.1. *For every prime $p > 0$ there exists a minimal surface of general type over an algebraically closed field of characteristic p that has a non-reduced Pic^0 .*

PROOF. In [Se, Proposition 15], Serre constructs for every $p > 0$ a smooth hypersurface Y_p in \mathbb{P}^3 with a fixed point free action of $C_p := \mathbb{Z}/p\mathbb{Z}$. From the construction it is clear that we may assume that Y_p is of degree ≥ 5 , i.e., of general type. Thus, the quotient $X_p := Y_p/C_p$ is a surface of general type with $h^1(\mathcal{O}_{X_p}) \neq 0$ by [Se, Proposition 16]. On the other hand, $b_1(Y_p) = 0$ implies $b_1(X_p) = 0$ since C_p acts without fixed points. Hence Pic^0 is not reduced. \square

Remark 3.2. Examples of uniruled surfaces of general type in characteristic 2 with arbitrary non-reduced Pic^0 have been constructed in [Lie1, Theorem 8.1].

Theorem 3.3. *Given an integer m , there exists an integer $P(m)$, such that minimal surfaces of general type with $K_X^2 = m$ over fields of characteristic $p \geq P(m)$ have a reduced Pic^0 .*

PROOF. Fixing K_X^2 , the Euler characteristic $\chi(\mathcal{O}_X) \leq 1 + p_g$ is bounded above by Noether's inequality and bounded below $\chi(\mathcal{O}_X) \geq 0$ in characteristic $p \geq 11$ by [S-B, Theorem 8]. Hence there is only a finite number of possibilities for $\chi(\mathcal{O}_X)$ if $p \geq 11$.

Canonical models of surfaces of general type with fixed $\chi(\mathcal{O}_X)$ and K_X^2 are parametrised by a subset of an appropriate Hilbert scheme which is defined over $\text{Spec } \mathbb{Z}$. Hence there exists a scheme \mathcal{M} of finite type over $\text{Spec } \mathbb{Z}$ and a family $f : \mathcal{X} \rightarrow \mathcal{M}$ of canonical models of surfaces of general type such that every such surface with $K^2 = m$ occurs in this family.

There exists an integer P_1 such that for every prime $p \geq P_1$ all components of \mathcal{M}_p are flat over $\text{Spec } \mathbb{Z}$. Let \mathcal{M}' be one of these finitely many components. By [Ar] there exists a quasi-finite morphism $\mathcal{N}' \rightarrow \mathcal{M}'$ and a family $f' : \mathcal{Y} \rightarrow \mathcal{N}'$ that resolves the singularities of f simultaneously.

Then, $\mathcal{N}' \otimes_{\mathbb{Z}} \mathbb{Q}$ is non-empty and parametrises smooth and minimal surfaces of general type in characteristic zero. By the Lefschetz principle, we may assume that the family f' is defined over the complex numbers. Then, by Ehresmann's fibration theorem, these surfaces are diffeomorphic, which implies that all of them have the same first Betti number b_1 . Hence h^{01} is constant in this family being equal to $b_1/2$ by Hodge theory. It follows that not only the Pic^0 of all fibres in this family over $\mathcal{N}' \otimes_{\mathbb{Z}} \mathbb{Q}$ are reduced but that also h^{01} is constant.

By upper semicontinuity there exists a closed subset $\mathcal{V} \subseteq \mathcal{N}'$ over which h^{01} of a fibre may jump. By Chevalley's theorem, the image of \mathcal{V} in $\text{Spec } \mathbb{Z}$ is a constructible set, i.e., closed or open since $\text{Spec } \mathbb{Z}$ is one-dimensional. However, by what we have just seen, this image avoids the generic point

of $\text{Spec } \mathbb{Z}$ and so this image is a proper closed subset. In particular, there exists a P'_2 , such that for every prime $p \geq P'_2$, the fibre \mathcal{N}'_p does not intersect with \mathcal{V} . Since $p \geq P_1$, for every field K of characteristic $p \geq \max(P_1, P'_2)$ and every morphism $\text{Spec } K \rightarrow \mathcal{N}'$ the fibre $\mathcal{Y}_K := \mathcal{Y} \times_{\mathcal{N}'} \text{Spec } K$ is a surface of general type that lifts to characteristic zero. Since $p \geq P'_2$ the lifted surface and \mathcal{Y}_K have the same h^{01} . Moreover, these two surfaces have the same b_1 by [K-U, Lemma 10.2] and it follows that $2h^{01} = b_1$ for \mathcal{Y}_K . In particular, $\text{Pic}^0(\mathcal{Y}_K)$ is reduced.

We choose $P(m)$ to be the maximum of P_1 and the P'_2 's for every of the finitely many components of \mathcal{M} . Then, every minimal surface of general type with $K^2 = m$ over a field K of characteristic $p \geq P(m)$ corresponds to a $\text{Spec } K$ -valued point of \mathcal{M} and we have already seen that all corresponding surfaces have a reduced Pic^0 . \square

The proof does not give an effective bound for $P(m)$. To find such bounds, a more detailed analysis is needed, which we now illustrate by determining the optimal $P(1)$ explicitly.

Proposition 3.4. *Minimal surfaces of general type with $K^2 = 1$ have a reduced Pic^0 over fields of characteristic $p \geq 7$. There do exist minimal surfaces of general type with $K^2 = 1$ and non-reduced Pic^0 over fields of characteristic 5.*

PROOF. By [Lie2, Proposition 1.1], such surfaces fulfill $1 \leq \chi(\mathcal{O}_X) \leq 3$, $p_g \leq 2$, $b_1 = 0$ and $h^{01} \leq 1$. Hence if $\chi(\mathcal{O}_X) = 3$ we necessarily have $p_g = 2$ and $h^{01} = 0$ and in particular the Pic^0 of such a surface is reduced.

In case $\chi(\mathcal{O}_X) = 1$ the Pic^0 is reduced in characteristic $p \geq 7$ by [Lie2, Corollary 2.6], which is one of the main results of this article. The first example of such a surface with non-reduced Pic^0 in characteristic 5 is due to Miranda [Mir], cf. also [Lie2, Section 5].

If $\chi(\mathcal{O}_X) = 2$ we either have $p_g = 1$ and $h^{01} = 0$, and such a surface has a reduced Pic^0 , or $p_g = 2$ and $h^{01} = 1$, in which case the surface has a non-reduced Pic^0 , since $b_1 = 0$. However, in this latter case there exists a μ_p -, or an α_p -torsor above X (depending on whether Frobenius acts bijectively or trivially on $H^1(\mathcal{O}_X)$), and arguing as in the proof of [Lie2, Theorem 2.4] we find that such surfaces can only exist in characteristic 2. \square

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